Cosmological sector for localized mass and spin in 2+1 dimensional topologically massive gravity

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Abstract

The cosmological sector to the full non-linear topologically massive gravity (TMG) is obtained for localized sources of mass m and spin σ besides the asymptotically spinning conical flat sector previously obtained [7]. In a small region near but outside the sources, the metric resembles the spinning conical flat metric but we find that the mass m creates a negative deficit angle of $3m$ as opposed to m. Furthermore, it is not possible to recover the results of pure Einstein gravity in the limit $\mu \to \infty$ unlike the flat sector.

I. Introduction

Gravity in 2+1 dimensions has attracted much interest in the last few years. In supergravity theory, a mechanism in 2+1 dimensions originally proposed by Witten [11], has been implemented as a possible solution to the cosmological constant problem [9]. The authors demonstrated that although unbroken supersymmetry prohibits the possibility of a cosmological constant in the vacuum, massive states, more precisely massive soliton states, cause the space-time to be conically flat and the generators of the supersymmetry, which cause the usual spectrum doubling, fail to exist. That work was done in the context of ordinary Einstein gravity (appropriately supersymmetrized). It is well known that Einstein gravity is trivial in 2+1 dimensions i.e there is no propagating graviton and matter free regions are flat. However, 2+1 dimensions allows one to include the parity violating gravitational Chern-Simons term [1, 10] and the gravitational field now becomes a non-trivial propagating massive field. Such an inclusion is in fact obligatory for a theory containing fermions and parity violation; it is automatically induced via 1 loop quantum corrections with strength $N/2$ where N is the number of species of fermions [12]. The results of [9] rested on the fact that exterior to the soliton the

space-time was conically flat. It becomes a compelling and interesting exercise to discern what would the soliton and the exterior space-time represent for the case of topologically massive gravity (TMG).

Many years ago, Vuorio [6] obtained two solutions to TMG in vacuum ($T_{\mu\nu} = 0$): a trivial flat space and a cosmological solution i.e. a homogenous space-time with constant curvature scalars. Localized sources can now be embedded in either of these two backgrounds. In the case of the flat background, solutions with localized mass m and spin σ have been obtained in the linearized [1] and in the non-linear theory [7] and gives rise in both cases to an asymptotic spinning conical space-time. As mentioned in [1], the cosmological solutions are not accessible in the linearized case and one is required to employ the full non-linear theory. Cosmological solutions have already been obtained in the non-linear theory using delta function sources [2]. It has been shown that delta function sources can be accomodated with torsion at the source [8] but leads to inconsistencies in the torsion-free TMG theory (see [7]). An alternative and more consistent approach is to use non-singular sources that are arbitrarily localized.

We demonstrate that non-singular localized mass and spin sources in the full non-linear TMG theory support a cosmological sector. In this sector, the spin source dominates the helical time structure of the metric at short distances outside the source and the mass creates a negative deficit angle of $3m$ instead of the value m found in the flat sector. Far from the sources, the metric resumes the form of the cosmological homogenous space-time found by Vuorio. The cosmological sector is a disjoint sector in that it is not possibe to recover pure Einstein gravity in the limit $\mu \to \infty$ where one would expect the topological term to vanish.

II. Field Equations

We begin our work by writing down the well known field equations for TMG with energy-momentum tensor $T_{\mu\nu}$. The Einstein field equations including a topological mass term is given by [1, 2] (in units where $8\pi G = 1$)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{\mu} C_{\mu\nu} = -\kappa^2 \, T_{\mu\nu} \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor , $R \equiv R_{\mu\nu} g^{\mu\nu}$ is the curvature scalar, and $C_{\mu\nu}$ is the 3 dimensional Weyl-Cotton tensor. We simplify the field equations (1) by choosing a rotationally symmetric, stationary metric. The most general form for such a metric is given by [3]

$$ds^2 = q^2 (\, dt + \psi(r)\, d\theta \,)^2 - e^{\phi(r)} \left(dr^2 + r^2 \, d\theta^2 \right) \qquad i = 1, 2 \tag{2}$$

To recover Vuorio's solution in the region exterior to the source we set $q = 1$ as in his work [6] (we will also show that the metric with $q = 1$ does support solutions with localized mass and spin). The functions $\psi(r)$ and $\phi(r)$ completely determine the metric. The scalar twist $\rho(r)$ is given by [7]

$$\rho = e^{-\phi} \, \frac{\psi^{'}(r)}{r}. \tag{3}$$

where $\psi^{'}(r) \equiv \dfrac{d\psi(r)}{dr}$. To avoid a delta function ρ, $\psi(r)$ is set to zero at $r = 0$ (see [7] for more details). The (0,0), (0,j) and (i,j) component field equations (1) are respectively [7]

$$\frac{3}{4}\rho^2 - \frac{1}{2}e^{-\phi}\nabla^2\phi + \frac{1}{\mu}\rho^3 - \frac{1}{2\mu}\hat{\nabla}^2\rho - \frac{1}{2\mu}\,\rho\, e^{-\phi}\nabla^2\phi = -T_{00} \tag{4}$$

$$\frac{\epsilon^{jk}e^{-\phi}}{2}\partial_k\left(\rho + \frac{3}{2\mu}\rho^2 - \frac{1}{2\mu}e^{-\phi}\nabla^2\phi\right) = -T_0^j \tag{5}$$

$$-\, e^{-\phi}\delta^{ij}\left(-\frac{\rho^2}{4} - \frac{1}{2\mu}\rho^3 + \frac{1}{2\mu}\hat{\nabla}^2\rho + \frac{1}{4\mu}\,\rho\, e^{-\phi}\nabla^2\phi\right) - \frac{1}{2\mu}\hat{\nabla}^i\hat{\nabla}^j\,\rho = -T^{ij}\,. \tag{6}$$

As in [7], we approach the problem of localized sources not by actually specifying $T_{\mu\nu}$ but by examining the metric dependent side of the field equations and drawing conclusions on the scalar twist ρ and the function ϕ if $T_{\mu\nu}$ were localized.

III. Solving the Field Equations: Cosmological Sector

When solving the field equations in the vacuum i.e. $T_{\mu\nu} = 0$, Vuorio obtains that the scalar twist $\rho(r)$ is a constant that can have two possible values (corresponding to two metrics): one is given by $\rho(r) = 0$ and $e^{-\phi}\nabla^2\phi = 0$ which describes a flat metric (Minkowski space) and the second is $\rho(r) = -2\mu/3$ and $e^{-\phi}\nabla^2\phi = 2\mu^2/9$ which describes a cosmological space-time. To include sources of mass m and spin σ in this cosmological space, we introduce two functions $M(r)$ and $S(r)$ such that

$$e^{-\phi}\nabla^2\phi = 2\mu^2/9 + M(r) \quad \text{ and } \quad \rho(r) = \frac{-2\mu}{3} + S(r) \tag{7}$$

where $M(r)$ and $S(r)$ are zero outside the sources. We will now see that $M(r)$ is proportional to the mass density and $S(r)$ to the spin source density.

When substituting ρ above into the T_{0j} "spin" equation (5) one observes that $\rho + 3\rho^2/2\mu = -S + 3S^2/2\mu$. As expected the constant part of $\rho(r)$ i.e. $-2\mu/3$, has disappeared and does not contribute to the spin source. Clearly, $\rho(r)$ has been replaced by $-S(r)$ in Eq. (5). In flat space, $\rho(r)$ was the total conserved spin source density (see [7]) and we see that in the cosmological case, it is $-S(r)$ that takes on that role.

The mass m is defined as the total energy (i.e. the volume integral of T_{00}) when the spin source density $S(r)$ is zero. It will be localized in a region from $r = 0$ to $r = \epsilon$ so that $M(r)$ and T_{00} are zero for $r > \epsilon$. Substituting the quantities in Eq. (7) into the T_{00} equation (4), one obtains

$$\begin{aligned} m &= \frac{1}{6}\int M(r) e^{\phi}\, d^2 r \\ &= \frac{\pi}{3}\int_0^{\epsilon}\left(\nabla^2\phi - \frac{2\mu^2}{9}e^{\phi}\right) r\, dr \\ &= \frac{\pi}{3}\; r\,\phi'\Big|_0^{\epsilon} - \frac{2\pi\mu^2}{27}\int_0^{\epsilon} e^{\phi}\, r\, dr. \end{aligned} \tag{8}$$

There are two terms to evaluate above. In the first term, the limit $r \to \epsilon$ can be obtained by matching the exterior to the interior solution at $r = \epsilon$. Outside the mass source ($r > \epsilon$), $e^{-\phi}\nabla^2\phi = 2\mu^2/9$ and the most general solution is given by [6]

$$\phi(r) = 2\ln\left(\frac{6n\, r^{n-1}}{\mu\, r_0^n\left[1 - \left(\dfrac{r}{r_0}\right)^{2n}\right]}\right) \quad \text{for } r > \epsilon \tag{9}$$

where n and r_0 are arbitrary constants and $r < r_0$. Note that $\phi(r)$ is invariant under the transformation $n \to -n$ so that positive and negative values for n are both valid (nonetheless, this invariance will have no direct physical consequences and for the sake of clarity and without loss of generality we can assume n is positive). With ϕ given above, one obtains

$$r\phi' = 2(n-1) + \frac{4n(r/r_0)^{2n}}{[1-(r/r_0)^{2n}]} \quad \text{for } r > \epsilon \tag{10}$$

and

$$\lim_{r\to\epsilon}(r\phi') = 2(n-1) \tag{11}$$

where ϵ is sufficiently small so that terms proportional to ϵ are negligible and have been dropped in the above limit. For reasons given in the introduction, we

do not allow delta function sources and therefore we exclude the possibility that $\phi(r) \propto \ln r$ as $r \to 0$. Therefore

$$\lim_{r\to 0}(r\phi^{'}) = 0 \tag{12}$$

i.e. if $\lim_{r\to 0}(r\phi^{'}) = k$ where $k \neq 0$, then $\phi(r) \propto k \ln r$ as $r \to 0$. With the above limits, Eqs. (11) and (12), the first term in Eq. (8) is

$$r\phi^{'}\Big|_0^{\epsilon} = 2(n-1). \tag{13}$$

The mass m is positive and therefore $n > 1$. To evaluate the second term in Eq. (8), the behaviour of $\phi(r)$ in the source region needs to be known. In the exterior, $\phi(r)$ is negative and decreases as $r \to \epsilon$ from the right i.e. behaves as $\ln r$ as $r \to \epsilon$. For an elementary localized particle the mass density $M(r)$ (and hence $\nabla^2\phi$) should be positive in the source region and this implies that $\phi(r)$ must continue to decrease from $r = \epsilon$ to $r = 0$. It follows that the quantity e^{ϕ} is of the order ϵ (or smaller) in the source region and that the second term in Eq. (8) can be neglected for ϵ sufficiently small. Substituting Eq. (13) into Eq. (8) one obtains

$$r\phi^{'}\Big|_0^{\epsilon} = \frac{3m}{\pi} \quad \text{and} \quad n = 1 + \frac{3m}{2\pi} \tag{14}$$

We now turn to spin. The spin source $S(r)$ will be localized in a small region from $r = 0$ to r_s where $r_0 \gg r_s \gg \epsilon$. By integrating the T_0^j equation (5)with $\rho(r)$ given by Eq. (7) one obtains

$$\int \epsilon^{ij}\, x^i \left(-T_0^j\right) e^{2\phi}\, d^2r = -\pi \int_0^{r_s} \left(-S(r) + \frac{3}{2\mu}S^2(r) - \frac{1}{2\mu}\, e^{-\phi}\nabla^2\phi\right)^{'} e^{\phi}\, r^2\, dr\,. \tag{15}$$

The integral of the third term can be readily evaluated using Eqs. (10) and (12) and yields $-\,(3m/\mu)\,(1 + 3m/4\pi)$ (where the second term in Eq. (10), proportional to $(r_s/r_0)^{2n}$, is negligible and has been dropped). For the integral of the first two terms we take $S(r)$ to be a rapidly decreasing function of r with the condition that $\lim_{r\to r_s} S(r)r^2e^{\phi} = \lim_{r\to r_s} S^2(r)r^2e^{\phi} = 0$. These integrals are well defined for any regular mass distribution $\nabla^2\phi$, but depend on the actual profile and there will only be small differences for any two well localized mass distributions. However, in the point mass limit the result is

$$(2\pi + 3m)\int_0^{r_s} \left(-S + \frac{3}{2\mu}S^2\right) e^{\phi} r dr\,. \tag{16}$$

Equation (15) can now be expressed as

$$J_S \equiv 2\pi \int_0^{r_s} -S(r) e^{\phi}\, r\, dr = \sigma + \frac{3m}{\mu}\left(\frac{4\pi + 3m}{4\pi + 6m}\right) - 2\pi \int_0^{r_s} \frac{3}{2\mu} S^2 e^{\phi} r dr\,. \quad (17)$$

where

$$\sigma \equiv \frac{2\pi}{2\pi + 3m} \int \epsilon^{ij}\, x^i \left(-T_0^j\right) e^{2\phi}\, d^2 r. \quad (18)$$

Here σ is identified as the bare spin i.e. it is equal to the spin source J_S when the topological term is absent ($\mu \to \infty$). The other two terms in Eq. (17) represent the induced spin (see [1, 7]).

We now proceed to find the metric Eq. (2). The function $\psi(r)$, given by Eq. (3) is

$$\begin{aligned} \psi(r) &= \int_0^r \rho(r) e^{\phi} r dr \\ &= \int_0^r S(r) e^{\phi} r dr - \frac{2\mu}{3} \int_0^r e^{\phi} r dr \\ &= \frac{-J_S}{2\pi} - \frac{12n\,(r/r_0)^{2n}}{\mu\,[1 - (r/r_0)^{2n}]} \qquad \text{for } r > r_s \end{aligned} \quad (19)$$

where we see that there are two distinct contributions to $\psi(r)$: one from the spin source density $S(r)$ and one from the "background" spin density $-2\mu/3$. In Eq. (19), the expression for ϕ exterior to the source Eq. (9), was used to evaluate the second integral. For $r < r_s$, $\psi(r)$ and hence the metric, are well behaved but depend intimately on the distribution of the mass and spin sources. Also, as mentioned in section II, $\psi(r)$ is zero at $r = 0$ and the metric is therefore nonsingular.

As in Vuorio's work [6] we define new variables

$$\tilde{\theta} = n\theta, \;\; \sinh x = \frac{2(r/r_0)^n}{1 - (r/r_0)^{2n}} \qquad \text{where } n = 1 + \frac{3m}{2\pi} \quad (20)$$

so that the metric, in terms of these new variables (dropping the tilde), is given by

$$ds^2 = \left[dt - \left(\frac{J_S}{(2\pi + 3m)} + \frac{6}{\mu}(\cosh x - 1)\right) d\theta\right]^2 - \frac{9}{\mu^2}\left(dx^2 + \sinh^2 x d\theta^2\right). \quad (21)$$

This metric is similar to Vuorio's "cosmological" metric [6] but differs from it in two ways: the constant n, which appears in the redefinition of θ, is not equal to 1 as in Vuorio's case and our metric has a non-zero spin J_S.

Since n is not 1, our metric has a deficit angle i.e. with $n = 1 + 3m/2\pi$ the new angle θ runs from 0 to $2\pi + 3m$ instead of 2π. Locally, this represents a conical space with negative angular defect of $3m$. In the case of the flat background the deficit angle was simply the mass m [1, 7]. The extra factor of three in the cosmological case is due to the spin-spin coupling term $\rho\, e^{-\phi}\nabla^2\phi/2\mu$ appearing in the T_{00} equation (4). This term contributes to the mass i.e. it is equal to $-e^{-\phi}\nabla^2\phi/3$ when m is being defined, that is when $S(r) = 0$ and ρ is $-2\mu/3$. The coupling is therefore between a "background" spin $\rho = -2\mu/3$ and an "induced" spin $e^{-\phi}\nabla^2\phi/2\mu$. In the flat case, the spin-spin term makes no contribution to the mass because there is no background spin. It is important to note that the part of the spin-spin term which survives in equation (4), $-M(r)/3$ is independent of μ and therefore it is impossible to make it vanish in the non-topological limit $\mu \to \infty$. Hence, the cosmological sector is a disjoint sector and one cannot recover pure Einstein gravity in the appropriate limit.

To see the effect of the spin J_S, note that in a region close to the source where x is small, the term with $\cosh x - 1$ in the metric is negligible compared to J_S and $\sinh^2 x \approx x^2$. In the neighbourhood of the source the metric is

$$ds^2 = \left[dt - \frac{J_S}{(2\pi + 3m)}\, d\theta\right]^2 - \frac{9}{\mu^2}\left(dx^2 + x^2 d\theta^2\right) \tag{22}$$

which is a spinning conical space with the spin J_S governing the helical- time structure of the metric(see [5, 6] for a discussion on helical- time structure). Far from the sources, the metric (21) of course describes the same space-time as Vuorio's i.e. a homogenous space-time with constant curvature scalars.

In conclusion, we have shown that there exists a distinct cosmologicial sector for the exterior space-time to localized spin and mass sources in 2+1 dimensional topologically massive gravity in addition to the usual flat, conical solution. This is in contra-distinction to ordinary Einstein gravity in 2+1 dimensions and also in 3+1 dimensions which admit only a unique exterior space-time. The ramifications of the existence of this sector should be investigated for the work of [9], for example.

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